Fourier Series

The Fourier series synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{2\pi jkt/T_0}$$

and analysis equation

$$X[k] = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-2\pi jkt/T_0} dt$$

say that $x(t)$ and $X[k]$ form a Fourier series pair. x is the time domain representation of the signal, and X is the frequency domain representation. They are really the same signal, just viewed from different perspectives. Unlike the case of Fourier transforms on $\mathbb{R}$, in this case the frequency domain is a different signal space than the time domain. The time domain is T_0-periodic functions on $\mathbb{R}$. The frequency domain is functions on $\mathbb{Z}$, the integers (the square brackets [] clearly indicate that it is a function of a discrete variable). This is a common theme in Fourier analysis: if one domain is periodic, the other is discrete.

In principle, we can always find the Fourier series (FS) pairs, if they exist, through direct evaluation of the analysis integral. Note that because $x(t)$ is periodic, it doesn't matter which period we integrate over, only that we integrate over exactly one period (e.g. $X[k] = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-2\pi jkt/T_0} dt$). Sometimes it is easier to use Poisson summation to find an FS from a similar Fourier transform.

Note that there is nothing preventing us from interpreting the discrete domain as the time domain and the periodic as the frequency domain. In this case, T_0 is usually fixed at 2π and the transform is called a Discrete-Time Fourier Transform (DTFT). We will study these as a special case of Z transforms.

In what follows, time domain is on the left, and frequency domain is on the right. Some transformation in one domain will have a corresponding transformation in the other domain.

There are rules for inversion (swapping x and X) and stretching in the frequency domain, but they are cumbersome to describe and I don't think they're worth learning.

- Fourier Series Pair

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} X[k] e^{2\pi jkt/T_0} \\
\frac{1}{T_0} \int_0^{T_0} x(t) e^{-2\pi jkt/T_0} dt &= X[k] \\
\mathcal{F}_{T_0}\{x(t)\} &= X[k] \\
x(t) &= \mathcal{F}_{T_0}^{-1}\{X[k]\} \\
x(t) &\Leftrightarrow X[k]
\end{aligned}$$

- Linearity ($\alpha, \beta \in \mathbb{C}$ are constant w.r.t. t and k)

$$\alpha x(t) + \beta y(t) \Leftrightarrow \alpha X[k] + \beta Y[k]$$

- Conjugation and Reflection
  - Complex Conjugation
  $$x^*(t) \Leftrightarrow X^\dagger[k]$$
  - Reflection
  $$x(-t) \Leftrightarrow X[-k]$$
  - Hermitian Conjugation
  $$x^\dagger(t) \Leftrightarrow X^*[k]$$
  - Symmetries
    * If $x(t)$ is purely real, then $X[k]$ is hermitian
    $$x(t) = x^*(t) \Longleftrightarrow X[k] = X^\dagger[k]$$
    * If $x(t)$ is purely imaginary, then $X[k]$ is antihermitian
    $$x(t) = -x^*(t) \Longleftrightarrow X[k] = -X^\dagger[k]$$
    * If $x(t)$ is even, then $X[k]$ is even
    $$x(t) = x(-t) \Longleftrightarrow X[k] = X[-k]$$
    * If $x(t)$ is odd, then $X[k]$ is odd
    $$x(t) = -x(-t) \Longleftrightarrow X[k] = -X[-k]$$
    * If $x(t)$ is hermitian, then $X[k]$ is purely real
    $$x(t) = x^\dagger(t) \Longleftrightarrow X[k] = X^*[k]$$
    * If $x(t)$ is antihermitian, then $X[k]$ is purely imaginary
    $$x(t) = -x^\dagger(t) \Longleftrightarrow X[k] = -X^*[k]$$

- Shifting[1]
  - Time Shift ($t_0 \in \mathbb{R}$ is a constant)
  $$x(t - t_0) \Leftrightarrow e^{-2\pi j k t_0/T_0} X[k]$$

[1]Note the different signs in the exponential.

- Modulation or Frequency Shift ($k_0 \in \mathbb{Z}$ is a constant)[2]

$$e^{2\pi j k_0 t/T_0} x(t) \Leftrightarrow X[k - k_0]$$

- Scaling ($m \in \mathbb{Z}$ is a constant)[3]

$$x(mt) \Leftrightarrow \begin{cases} X\left[\frac{k}{m}\right] & m|k \\ 0 & \text{else} \end{cases}$$

- Differentiation
  - Time Differentiation

$$\frac{d}{dt}x(t) \Leftrightarrow \frac{2\pi jk}{T_0}X[k]$$

- Convolution
  - Time Cyclic Convolution[4]

$$(x * h)(t) = \int_0^{T_0} x(\lambda)h(t-\lambda)d\lambda \Leftrightarrow T_0 X[k]H[k]$$

  - Time Multiplication or Frequency Convolution[5]

$$x(t)h(t) \Leftrightarrow (X * H)[k] = \sum_{m=-\infty}^{\infty} X[m]H[k-m]$$

  - Time Cyclic Cross-correlation

$$(x \star h)(t) = (x^\dagger * h)(t) = \int_0^{T_0} x^*(\lambda)h(t+\lambda)d\lambda \Leftrightarrow T_0 X^*[k]H[k]$$

  - Time Cyclic Autocorrelation

$$(x \star x)(t) = (x^\dagger * x)(t) = \int_0^{T_0} x^*(\lambda)x(t+\lambda)d\lambda \Leftrightarrow T_0|X[k]|^2$$

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[2]Note that we could not allow an arbitrary shift in $X[k]$ (because it's only defined on the integers), nor could we allow an arbitrary modulation of $x(t)$ (because it must remain T_0-periodic.

[3]The operation on $X[k]$ is also called *decimation* (cf. the Roman military punishment). The notation $m|k$ means "m divides k". For example, 3 divides 6 but not 4. What happens in the frequency domain is that the spectrum is streched out, and zeros are inserted. Every for each k, $X[k]$ is separated by $m-1$ zeros.

[4]The convolution is refered to as "cyclic" because x and h are periodic. Usually a regular convolution (with limits of integration at $\pm\infty$) is undefined for periodic functions.

[5]Note that the discrete convolution doesn't have a T_0 factor like the frequency multiplication rule does. A mnemonic is *convolution gets the constant.*

- Parseval's Identities or Unitarity[6]
  - Inner Product

$$\frac{1}{T_0}\int_0^{T_0} x(t)y^*(t)dt = \sum_{k=-\infty}^{\infty} X[k]Y^*[k]$$

  - Energy or Norm

$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X[k]|^2$$

- Area under the curve
  - DC or Average Value

$$\frac{1}{T_0}\int_0^{T_0} x(t)dt = X[0]$$

  - Value at $t = 0$

$$x(0) = \sum_{k=-\infty}^{\infty} X[k]$$

- Radian Frequency ($\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$)
  - Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j\omega_0 kt}$$

  - Analysis

$$X[k] = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$

  - Equivalence: Unlike the Fourier transform, there is no difference in the frequency domain representation of the signal when using radian frequency or regular (Hertzian) frequency.

[6]"Unitary" means "preserves inner products (and norms)."

- Poisson Summation
  - $x(t)$ is an aperiodic function that is zero outside of some interval of length T_0.
  - $X(f) = \mathcal{F}\{x(t)\}$ is the Fourier transform of $x(t)$.
  - $g(t) = \sum_{n=-\infty}^{\infty} x(t-nT_0)$ is a T_0-periodic function made up of translates of x.
  - $G[k] = \mathcal{F}_{T_0}\{g(t)\}$ is the discrete spectrum (Fourier series coefficients) of g.

$$g(t) = \sum_{n=-\infty}^{\infty} x(t-nT_0) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X\left(\frac{k}{T_0}\right) e^{2\pi jkt/T_0} = \sum_{k=-\infty}^{\infty} G[k] e^{2\pi jkt/T_0}$$